4.7 – Change of Basis

Note: Recall that $(\mathbf{v})_B$ is the coordinate vector for a vector $\mathbf{v}$ relative to a basis B. When considered as a column vector, this author uses the notation $[\mathbf{v}]_B$.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$ is the coordinate vector of $\mathbf{v}$ relative to S, then the mapping $\mathbf{v} \to (\mathbf{v})_S$ is the **coordinate map relative to** S from V to R^n.

To transition from one basis B to another basis B' for the same vector space, we use a transition matrix, denoted by $P_{B \to B'}$.

In 4.5, we found $(\mathbf{v})_B$ for $\mathbf{v} = (-1,\ 7,\ 2)$ with respect to the basis $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2,\ -1,\ -1)$, $\mathbf{v}_2 = (-2,\ 1,\ 2)$, and $\mathbf{v}_3 = (3,\ 5,\ 4)$ by row reducing an augmented matrix. Here we achieve the same result by using a transition matrix from S, the standard basis for R^3, to B.

Definition: If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1,\ \mathbf{u}'_2,\ \ldots,\ \mathbf{u}'_n\}$, then $[\mathbf{u}_i]_{B'}$ are the coordinate vectors for the old basis vectors relative to the new basis. The **transition matrix from B to B′** is the matrix having these vectors as its columns and is written [by this author] as $P_{B \to B'} = [[\mathbf{u}_1]_{B'} | [\mathbf{u}_2]_{B'} | \ldots | [\mathbf{u}_n]_{B'}]$. Note that this is a partitioned matrix.

We use this as follows: For every vector $\mathbf{v}$ in V,

$$[\mathbf{v}]_{B'} = P_{B \to B'} [\mathbf{v}]_B$$

#**3** Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \left\{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\right\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix B to B'.

b. Compute the coordinate vector $[\mathbf{w}]_B$, where $\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ and use the transition matrix in part (a) to compute $[\mathbf{w}]_{B'}$.

c. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

Theorem **4.7.1** If P is the transition matrix from a basis B to a basis B' for a finite-dimensional vector space V, then P is invertible, and P^{-1} is the transition matrix from B' to B.

#13 If P is the transition matrix from a basis B' to a basis B, and Q is the transition matrix from B to a basis C, what is the transition matrix from B' to C? What is the transition matrix from C to B'?